J. Appl. Maths Mechs Vol. 57, No. 3, pp. 455–460, 1993
Printed in Great Britain.

0021–8928/93 $24.00+.00

CALCULATION OF THE FLOW INDUCED BY INJECTING GAS THROUGH THE WALLS OF A CHANNEL OF FINITE LENGTH†

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(*Received* 3 *December* 1991)

A method is proposed for calculating the flow of a gas in a narrow planar channel of finite length. The gas is injected through the channel walls and effuses into the surrounding medium at a specified pressure. The calculations are carried out in the Prandtl approximation. A special feature of the method being proposed is the possibility of determining the flow characteristics for various values of the characteristic Mach number on the basis of the results of a single calculation carried out at a fixed Reynolds number. The procedure for taking account of the ellipticity of the problem being considered is considerably simplified.

FLOWS induced by injecting a gas through the walls of channels of finite length have been modelled (see [1, 2], for example) using certain considerable simplifications. The flow in a closed planar channel, when there is evaporation from and condensation on one of its walls and a recirculation zone, have been investigated in [3] by solving the Navier–Stokes equations.

1. A planar flow in a channel of length $2l^*$ and width $2h^*$ is considered. The flow is induced by injecting a gas through the permeable walls at a constant flow rate density J^* and gas temperature T_0^*. The gas effuses through the open ends of the channel into the surrounding space in which the pressure p_e^* is specified (dimensional quantities are denoted by an asterisk). The main interest lies in finding the pressure distribution along the channel as a function of the controlling parameters of the problem, l^*, h^*, J^*, p_e^* and T_0^*.

A rectangular system of coordinates with the origin located in the middle of one wall (the lower one) of the channel, with the x^*-axis directed along the channel and the y-axis perpendicular to its walls, is introduced for the numerical solution of the problem. When the condition $h^*/l^* \ll 1$ is satisfied, the flow in the channel may be calculated in the Prandtl approximation [4].

If the controlling parameters of the problem such as the half length of the channel, l^*, and the pressure of the surrounding medium p_e^*, are employed as characteristic values to represent the equations in dimensionless form, then the characteristic Mach number, $M_0 = (R^* T_0^* / \gamma)^{1/2} J^* l^* / (p_e^* h^*)$, where R_e^* and γ are the gas constant and the adiabatic index, appears in the momentum and energy equations for the pressure-gradient terms.

Moreover, to solve the parabolic Prandtl equations, it is necessary to specify initial conditions at the centre of the channel $x = 0$. These can be obtained from the solution of a boundary-value problem for the system of ordinary differential equations which are obtained when the Prandtl equations are expanded in the neighbourhood of $x = 0$. Previously unknown values of the pressure at the centre of the channel and its second derivative with respect to x

†*Prikl. Mat. Mekh.* Vol. 57, No. 3, pp. 50–56, 1993.

enter into this system of equations. The second derivative with respect to x can be determined, as is shown below, by satisfying one of the two boundary conditions for the vertical velocity. In order to determine the value of the pressure at $x=0$, an algorithm is required which ensures that the effect of the external pressure p_e^*, on the pressure in the channel, under "subcritical" conditions of escape from it, is taken into account. For example, the pressure at the centre of the channel can be determined by some iterative method by specifying its value, carrying out a calculation up to the outlet cross-section of the channel and verifying the condition that the values of the pressure in the outlet cross-section and p_e^* are identical under "subcritical" conditions. Under "supercritical" outflow conditions, the external pressure has no effect on the flow in the channel and, in this case, it is necessary to check that the "critical" conditions are satisfied in the outlet cross-section. What is meant by the term "critical" conditions in this paper will be explained later.

The solution of the problem under consideration is considerably simplified if, in writing out the equations in dimensionless form, the pressure at the centre of the channel, p_0^*, which will be determined after solving a boundary-value problem, is used as the characteristic value of the pressure. In this case, as will be shown below, it is much simpler to take account of the effect of the ambient pressure under "subcritical" conditions, and the pressure distribution under "supercritical" conditions is determined just as simply.

On representing the product of the density of the gas by the vertical velocity in the form $\rho^* v^* = J^* V$, where V is an unknown function and choosing the quantity $\rho_0^* = p_0^*/(R^* T_0^*)$ as the characteristic value of the density, using the equation of continuity we obtain an expression for the characteristic value of the longitudinal flow velocity $u_0^* = J^* L^* R^* T_0^*/(p_0^* h^*)$. Here L^* is an effective length which is of the order of the half length of the channel l^* and is introduced in such a manner that the equality $p_0^*/(\rho_0^* u_0^{*2}) = 1$ is satisfied. The expression

$$L^* = \frac{p_0^* h^*}{J^* \sqrt{R^* T_0^*}} \tag{1.1}$$

follows from this.

The characteristic length L^* was introduced with the aim of eliminating the Mach number from the equations. In dimensionless coordinates $x = x^*/L^*$ and $y = y^*/h^*$, the system of Prandtl equations with the corresponding boundary conditions has the form

$$\frac{p}{T} u \frac{\partial u}{\partial x} + V \frac{\partial u}{\partial y} + \frac{dp}{dx} = \frac{1}{\mathrm{Re}} \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{1.2}$$

$$\frac{p}{T} u \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} + \frac{1-\gamma}{\gamma}\left[u \frac{dp}{dx} + \frac{\mu}{\mathrm{Re}}\left(\frac{\partial u}{\partial y}\right)^2\right] = \frac{1}{\mathrm{Re}\,\mathrm{Pr}} \frac{\partial}{\partial y}\left(\mu \frac{\partial T}{\partial y}\right)$$

$$V = \int_y^1 \frac{\partial}{\partial y}\left(\frac{pu}{T}\right) dy \quad \left(\mathrm{Re} = \frac{J^* h^*}{\mu_0^*}, \quad \mathrm{Pr} = \frac{c_p^* \mu_0^*}{\eta_0^*}\right)$$

$$y = 0; \quad u = 0, \quad T = 1, \quad V = 1 \tag{1.3}$$

$$y = 1: \quad \frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = 0 \tag{1.4}$$

Here c_p^* is the heat capacity of the gas and μ_0^* and η_0^* are the coefficient of viscosity and the thermal conductivity at a temperature T_0^*. In writing out Eq. (1.2), the equation of state of an ideal gas was used. The boundary conditions (1.4) are symmetry conditions which hold when the gas is injected through both walls of the channel at the same intensity. The last condition in (1.3) is used to determine the pressure gradient dp/dx at each specified cross-section of the channel.

The initial conditions for the parabolic equations (1.2) are found from the solution of the boundary-value problem obtained by expanding the required functions in the neighbourhood of $x=0$

$$u = xU(y) + \dots, \quad V = V(y) + \dots, \quad T = T(y) + \dots, \quad dp/dx = xP + \dots \tag{1.5}$$

As a result of the substitution of (1.5) into (1.2)–(1.4), we obtain the following boundary-value problem

$$\begin{aligned} &\frac{U^2}{T} + VU' + P = \frac{1}{\mathrm{Re}}(\mu U')', \quad VT' = \frac{1}{\mathrm{RePr}}(\mu T')', \quad V = \int_y^1 \frac{U}{T}dy \\ &y = 0: \; U = 0, \quad T = 1, \quad V = 1; \quad y = 1: \; U' = T' = 0 \end{aligned} \tag{1.6}$$

Differentiation with respect to y is indicated by a prime.

There are five boundary conditions for the fourth-order systems of Eqs (1.6) in the unknown functions U and T. One of them, namely, $V(0)=1$, is used to determine the quantity $P=d^2p(0)/dx^2$ occurring in (1.6) which is found by the secant method during the process of the iterative solution of the boundary-value problem (1.6). There are no other undetermined parameters in this problem. If the pressure is referred to p_e^*, another unknown quantity $p(0)$, apart from P, occurs in the equations, analogous to (1.6). Since the number of boundary conditions is the same here, it is necessary to use the matching conditions in the outlet cross-section, about which we have spoken above, to determine $p(0)$.

When $\mathrm{Pr}=\mathrm{const}$ the values of the parameter P are solely dependent on the Reynolds number. This dependence of the quantity $-P$ on Re when $\mathrm{Pr}=1$ is shown in Fig. 1.

The profiles $U(y)$, $V(y)$ and $T(y)$ and the quantity P, obtained from the solution of problem (1.6), taking account of expansions (1.5), are used as the initial conditions in solving problem (1.2)–(1.4). The pressure in the channel $p(x)$ is determined by integration of the gradient dp/dx with respect to x and the initial value in the integration $p(0)=1$.

2. During the course of the solution of the boundary-value problem (1.2)–(1.4), which is carried out by a numerical method of second order of accuracy with respect to both variables [5], the mean velocity coefficient $\langle\lambda\rangle$ is found in each specified cross-section x.

The following averaging procedure [6] is used: at each point of the difference grid with respect to y in the cross-section under consideration, the local Mach number $M=u/\sqrt{(\gamma T)}$, the velocity coefficient $\lambda = M\{(\gamma+1)[2+(\gamma-1)M^2]\}^{1/2}$ and the stagnation temperature $T_t = T/\tau(\lambda)$, where $\tau(\lambda) = 1-\lambda^2(\gamma-1)/(\lambda+1)$, were determined using the calculated velocity profile $u(x, y)$, and the temperature profile $T(x, y)$.

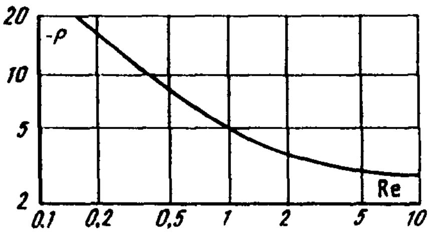

FIG. 1.

The mean stagnation temperature is determined from the condition of conservation of the flow energy during averaging

$$\langle T_t\rangle = \int_0^1 Y(\lambda)T_t^{1/2}dy\left(\int_0^1 Y(\lambda)T_t^{-1/2}dy\right)^{-1}, \quad Y(\lambda)=\left(\frac{\gamma+1}{2}\right)^{1/(\gamma-1)}\frac{\lambda}{\tau(\lambda)}$$

The mean value of the velocity of sound is then calculated

$$\langle a_{cr}^*\rangle = \left(\frac{2\gamma}{\gamma+1}R^*T_0^*\langle T_t\rangle\right)^{1/2}$$

The condition of conservation of momentum during averaging serves to determine the mean velocity coefficient

$$\frac{\gamma+1}{2\gamma}Q^*\langle a_{cr}^*\rangle z(\langle\lambda\rangle) = h^*p^*\int_0^1 j(\lambda)dy, \quad z(\lambda)=\lambda+\frac{1}{\lambda}, \quad j(\lambda)=\frac{1+\lambda^2}{\tau(\lambda)} \tag{2.1}$$

$$Q = \int_0^h \rho^* u^* dy^*$$

where Q^* is the flow rate in the cross-section of the channel under consideration.

By expressing the flow rate in terms of the calculated profiles $u(x, y)$ and $T(x, y)$ and substituting the resulting expression into (2.1), we finally obtain the equation

$$\langle\lambda\rangle+\frac{1}{\langle\lambda\rangle} = \left(\frac{2\gamma}{\gamma+1}\frac{1}{\langle T_t\rangle}\right)^{1/2}\left(\int_0^1 \frac{u}{T}dy\right)^{-1}\int_0^1 j(\lambda)dy \tag{2.2}$$

Here, the root $\langle\lambda\rangle \leqslant 1$ is taken.

Problem (1.2)–(1.5) is solved up to the cross-section x_m, in which $\langle\lambda\rangle = 1$.

The pressure distribution along the channel, corresponding to a definite value of its half length l^*, is found in the following manner. A value of the coordinate $x_l = l^*/L^*$, which when account is taken of (1.1), is determined by the expression

$$x_l = \frac{J^*l^*}{p_0^*h^*}(R^*T_0^*)^{1/2} \tag{2.3}$$

in which the as yet unknown quantity p_0^* occurs, corresponds to the outlet cross-section of the channel.

It is next assumed that, if the value of $\langle\lambda\rangle$ determined from (2.2) is less than unity in the outlet cross-section of the channel, the pressure in this cross-section is equal to the external pressure: $p^*(l^*) = p_e^* = p_0^* p_l$, where $p_l = p(x_l)$ is the calculated value. Whereupon, by substituting the value of p_0^* into (2.3), we obtain an equation for the coordinate of the outlet cross-section of the channel

$$x_l = \sqrt{\gamma}M_0 p(x_l), \quad M_0 = \frac{p_*^*}{p_e^*}, \quad p_*^* = \frac{J^*l^*}{h^*}\left(\frac{R^*T_0^*}{\gamma}\right)^{1/2} \tag{2.4}$$

Here we have introduced an effective Mach number M_0, which depends on the ambient pressure p_e^* which does not occur among the parameters of problem (1.2)–(1.6). The physical meaning of the quantity M_0 will be explained below.

As follows from (2.4), a decrease in the value of p_e^* at constant J^*, h^*, l^* and T_0^* leads to an increase in x_l and this occurs so long as $x_l < x_m$. A value of $\langle\lambda\rangle = 1$ is attained in the outlet cross-section of the channel when the Mach number reaches a value $M_{0m} = x_m/[\sqrt{(\gamma)}p(x_m)]$. It is assumed that, when $M_0 \geqslant M_{0m}$ (M_{0m} is a function of the Re number), critical conditions of "blocking" of the channel are established, under which a change in the ambient pressure has no effect on the flow in the channel. In this case the value x_m corresponds to the coordinate of the outlet cross-section of the channel and the pressure at its centre is determined from (2.3)

$$p_0^* = \frac{J^* l^*}{x_m h^*}(R^* T_0^*)^{1/2} = p_e^* \frac{\sqrt{\gamma}}{x_m} M_0 \tag{2.5}$$

Since, from the conditions of the problem, the ambient pressure is an unknown quantity, the pressure in the channel must be expressed in terms of p_e^*. When $M_0 \leqslant M_{0m}$, x_l is found from (2.4) and the pressure distribution in the channel referred to p_e^*, $p'(x') = p(x)/p_l$, where $x' = x/x_l$, is constructed on the basis of the distribution $p(x)$ obtained from the solution of (1.2)–(1.6). When $M_0 \geqslant M_{0m}$, it follows from (2.5) that $p'(x') = p(x)\sqrt{(\gamma)}M_0/x_m$, where $x' = x/x_m$. Let us now obtain an estimate of the value of the mean Mach number in the outlet cross-section of the channel

$$\langle M\rangle = \langle u^*\rangle(\gamma R^* \langle T^*\rangle)^{-1/2}\left(\langle u^*\rangle = \frac{1}{h^*}\int_0^h u^*(l^*, y^*)dy^*\right)$$

Since the temperature across the channel varies insignificantly, the estimate

$$Q^* = \frac{p_l^*}{R^*}\int_0^h \frac{u^*}{T^*}dy^* \cong \frac{p_l^* h^*}{R^* T_0^*}\langle u^*\rangle$$

holds.

On the other hand, $Q^* = J^* l^*$, from which, when account is taken of the relationship $\langle T^*\rangle \cong T_0^*$, we obtain $\langle M\rangle \cong M_0 p_e^*/p_l^*$. Hence, when $M_0 \leqslant M_{0m}$, the characteristic Mach number M_0 is approximately equal to the average value of the Mach number in the outlet cross-section.

3. The basic advantage of this method is therefore the possibility of determining the flow characteristics in a channel (the pressure distribution, for example) for different values of the characteristic number M_0 (or the ambient pressure) from the results of a single calculation carried out at a fixed Re number.

The results of the calculation of the pressure distribution in a channel, $p(x)$, at the constant values $T_0^* = 300$ K and $\Pr = 1$ are shown in Fig. 2 for three values of the Reynolds number: $\text{Re} = 0.1$, 1 and 10 (curves 1, 2 and 3, respectively). The values of the limiting coordinate x_m are indicated by small arrows on the x-axis.

The coordinate of the outlet cross-section x_l, determined from Eq. (2.4) (the solid curves) and the pressure in this cross-section p_l (the dashed line curves) as a function of M_0 are shown in Fig. 3. The distributions in Fig. 2 were used in finding the value of x_l. The numbering of the curves in Fig. 3 is the same as that in Fig. 2.

It is seen that the value of x_l increases as the Reynolds number increases. At the same time, the pressure at the centre of the channel $p_0^* = p_e^*/p_e$ also increases and, consequently, the pressure drop in the channel increases.

The dependences of the coordinate of the outlet cross-section under "blocking" conditions, x_m, the pressure $p_m = p(x_m)$ and the limiting value of the Mach number M_{0m} at which these conditions are established are shown in Fig. 4 as a function of the Reynolds number.

Conclusions can be drawn on the basis of the above results regarding the qualitative

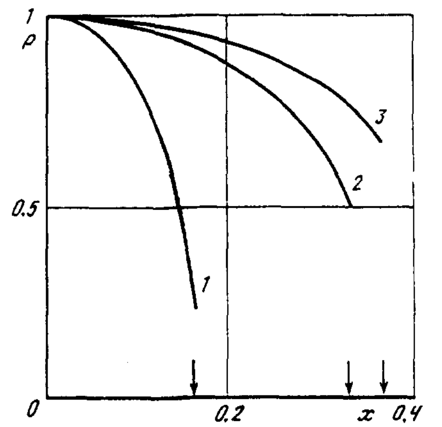

FIG. 2.

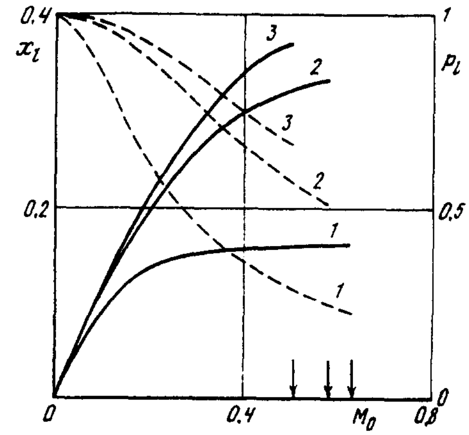

FIG. 3.

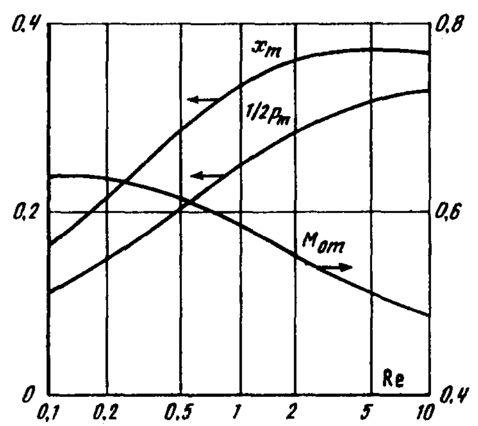

FIG. 4.

dependences of the flow characteristics in a channel on the dimensional parameters of the problem, l^*, h^*, J^*, p_e^* and T_0^*.

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Translated by E.L.S.